Spectral Methods in Time for Parabolic Problems

Hillel Tal-Ezer

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SPECTRAL METHODS IN TIME FOR PARABOLIC PROBLEMS*

HILLEL TAL-EZER†

Abstract. A pseudospectral explicit scheme for solving linear, periodic, parabolic problems is described. It has infinite accuracy both in time and in space. The high accuracy is achieved while the time resolution parameter M ($M = O(1/\Delta t)$ for time marching algorithm) and the space resolution parameter N ($N = O(1/\Delta x)$) must satisfy $M = O(N^{1+\varepsilon})$ $\varepsilon > 0$, compared to the common stability condition $M = O(N^2)$, which must be satisfied in any explicit finite-order time algorithm.

Key words. spectral methods, explicit scheme, parabolic problems, Chebyshev expansion

AMS(MOS) subject classification. 65M99

1. Introduction. In recent years, it has been shown that spectral methods can provide a very useful tool for the solution of time-dependent partial differential equations [3]. A standard scheme uses spectral methods to approximate the space derivatives and a finite difference approach to march the solution in time. This tactic results in an unbalanced scheme; it has infinite accuracy in space and finite accuracy in time. It is obvious that the overall accuracy is influenced strongly by the relatively poor approximation of the time derivative. Moreover, using finite-order explicit scheme results in a very stringent stability condition. The timestep Δt must satisfy

$$\Delta t = O\left(\frac{1}{N^2}\right) \tag{1.1}$$

where N is the number of grid points in space. This severe condition is commonly overcome by resorting to implicit schemes. Varga [6] and Cody, Meinardus, and Varga [2] approached these problems by using Chebyshev rational approximations of the evolution operator. Thus, they overcome two drawbacks: low accuracy and stringent stability condition. In fact, the implicit scheme presented in [2], [6] is unconditionally stable, and the error in time decays exponentially.

Implicit algorithms involve inverting matrices. When the space approximation is based on finite differences or finite elements (as in [2], [6]), the related matrices are banded ones (e.g., tridiagonal), which makes them relatively easy to invert. On the other hand, using spectral methods for the space discretization results in full matrices. Inverting these matrices is a time-consuming procedure.

In this article we describe an explicit scheme for the solution of parabolic problems when the space discretization is done by spectral methods. This scheme is highly efficient (its efficiency is equivalent to having a stability condition $\Delta t = O(1/N)$) and the error in time decays exponentially. In § 2 we present a model problem and its fully discrete solution. The new approach for approximating the evolution operator is described in § 3. In § 4 we carry out an error and stability analysis. Numerical experiments confirming the theoretical results are presented in § 5.

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† School of Mathematical Sciences, Tel-Aviv University, Ramat Aviv, 69978 Tel Aviv, Israel. Current address, Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912.

2. The model problem. Let us consider the heat equation

$$\begin{aligned} &U_t - GU = 0, \qquad 0 < x < \Pi, \\ &U(x, 0) = U^0(x), \\ &U(0, t) = U(\Pi, t) = 0, \end{aligned} \tag{2.1}$$

where G is the spatial operator

$$G = a\frac{\partial^2}{\partial x^2}. \tag{2.2}$$

Discretizing (2.1) in space using pseudospectral Fourier method, we obtain a semi-discrete representation

$$\begin{aligned} &(U_N)_t - G_N U_N = 0, \\ &U_N(x, 0) = U_N^0(x), \\ &U_N(0, t) = U_N(\Pi, t) = 0 \end{aligned} \tag{2.3}$$

while

$$U_N = P_N U, \quad G_N = P_N G P_N, \quad U_N^0 = P_N U^0 \tag{2.4}$$

and where for any function $f(x)$, $P_N f(x)$ is its sine interpolant at the collocation points

$$x_j = j\Pi/N, \qquad j = 0, 1, \cdots, N-1, \tag{2.5}$$

or, more precisely,

$$P_N f(x) = \sum_{k=0}^{N-1} a_k \sin(kx) \tag{2.6}$$

where

$$a_k = \frac{2}{N} \sum_{j=0}^{N-1} f(x_j) \sin(kx_j). \tag{2.7}$$

G_N is an operator defined on N-dimensional subspace; thus, it can be represented as a $N \times N$ matrix. The formal solution of (2.3) is

$$U_N(x, t) = \exp(tG_N) U_N^0(x) \tag{2.8}$$

where $\exp(tG_N)$ is the exact evolution operator. A fully discrete solution of (2.1) is achieved by approximating this evolution operator. In [5], it has been shown that any explicit time scheme can be represented as

$$V_N^M = H_M(tG_N) U_N^0 \tag{2.9}$$

where $H_M(z)$ is a polynomial of degree M that converges to e^z in the domain that includes all the eigenvalues of the operator tG_N. V_N^M is the fully discrete solution and $H_M(tG_N)$ is the numerical evolution operator.

3. The orthogonal polynomials scheme. Let E be the error that results from approximating the evolution operator. Then

$$E = [\exp(tG_N) - H_M(tG_N)] U_N^0. \tag{3.1}$$

The eigenvectors of the matrix tG_N are $W_1, \cdots, W_N$, where $(W_k)_j = \sin(kx_j)$. Due to the orthogonality of this set of eigenvectors, tG_N is a normal matrix and there is an orthogonal matrix S_N such that

$$E = S_N D_N S_N^{-1} U_N^0 \tag{3.2}$$

while D_N is the diagonal matrix

$$(D_N)_{kk} = e^{\lambda_k t} - H_M(\lambda_k t) \tag{3.3}$$

and $\lambda_k t$ are the eigenvalues of tG_N. Since S_N is an orthogonal matrix, we have $\|S_N\| = \|S_N^{-1}\| = 1$. Therefore,

$$\|E\|_{L_2} \leqq \|S_N\|_{L_2} \|D_N\|_{L_2} \|S_N^{-1}\|_{L_2} = \|D_N\|_{L_2}$$

or

$$\|E\|_{L_2} \leqq \max_{z \in I} |e^z - H_M(z)| \tag{3.4}$$

where I is the domain that includes all the eigenvalues of tG_N. In our case,

$$I = [-aN^2 t, 0]. \tag{3.5}$$

A standard finite-order scheme can be characterized by a polynomial $H_M(z)$ based on a Taylor expansion of e^z. Thus, it has high accuracy only for a small z. The error increases rapidly when z is increased. This property explains the poor accuracy and stringent stability condition mentioned in the Introduction.

Let us take, for example, the modified Euler scheme. The numerical evolution operator is

$$H_M(tG_N) = (I + \Delta t G_N + \tfrac{1}{2}(\Delta t G_N)^2)^n \tag{3.6}$$

where

$$\Delta t = t/n. \tag{3.7}$$

Thus,

$$H_M(z) = \left(1 + \frac{1}{n} z + \frac{1}{2n^2} z^2\right)^n \qquad (M = 2n) \tag{3.8}$$

and

$$[H_M(z)]^{1/n} = 1 + \frac{1}{n} z + \frac{1}{2n^2} z^2. \tag{3.9}$$

Equation (3.9) is the first three terms of Taylor expansion of $e^{z/n}$. Observing Fig. 1, we find that $H_M(z)$ converges to e^z when

$$-2n \leqq z \leqq 0. \tag{3.10}$$

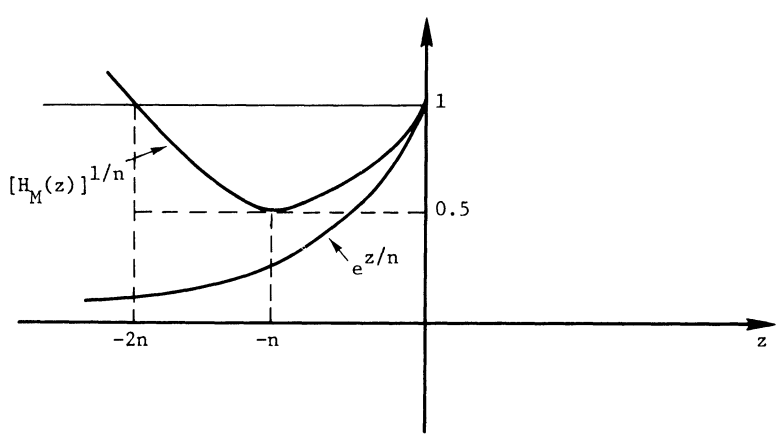

FIG. 1

(For accuracy, a more stringent condition is necessary.) Using (3.5), (3.7), and (3.10) results in the following stability condition:

$$\Delta t \leqq \frac{2}{a}\left(\frac{1}{N^2}\right). \tag{3.11}$$

Expression (3.4) suggests that a uniform approximation of e^z is preferable. Such an approximation is achieved when we use Chebyshev polynomial expansion of the exponential function (see the discussion in [5] for hyperbolic problems). Let

$$w = \frac{1}{R}(z+R), \qquad -1 \leqq w \leqq 1 \tag{3.12}$$

where

$$R = \tfrac{1}{2}aN^2 t. \tag{3.13}$$

It then follows that

$$e^z = e^{-R}\, e^{Rw} = \sum_{k=0}^{\infty} b_k T_k(w) \tag{3.14}$$

where $T_k(w)$ is the Chebyshev polynomial of order k and [1]

$$b_k = e^{-R} c_k \int_{-1}^{1} e^{Rw} T_k(w)(1-w^2)^{-1/2}\, dw = e^{-R} c_k I_k(R) \tag{3.15}$$

and

$$c_k = \begin{cases} 1, & k = 0, \\ 2, & k \geqq 1. \end{cases} \tag{3.16}$$

$I_k(R)$ is the modified Bessel function of order k. Thus, the M degree polynomial approximation of e^z is

$$H_M(z) = \sum_{k=0}^{M} b_k T_k(w(z)). \tag{3.17}$$

Because of (3.12), we substitute the operator F_N defined as

$$F_N = \frac{1}{R}[tG_N + RI] \tag{3.18}$$

for w. $H_M(F_N)$ is the numerical evolution operator. Thus, the fully discrete numerical solution of (2.1) is

$$V_N^M = H_M(F_N) U_N^0 = \sum_{k=0}^{M} b_k T_k(F_N) U_N^0. \tag{3.19}$$

$T_k(F_N) U_N^0$ is computed by using the recurrence relation

$$\begin{aligned} &T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x), \qquad k \geqq 2, \\ &T_0(x) = 1, \qquad T_1(x) = x. \end{aligned} \tag{3.20}$$

Hence,

$$\begin{aligned} &T_k(F_N) U_N^0 = 2F_N T_{k-1}(F_N) U_N^0 - T_{k-2}(F_N) U_N^0, \qquad k \geqq 2, \\ &T_0(F_N) U_N^0 = U_N^0, \qquad T_1(F_N) U_N^0 = F_N U_N^0. \end{aligned} \tag{3.21}$$

The algorithm defined by (3.19), (3.21) can be regarded as a three-level scheme since it uses the recurrence relation. Therefore, it has the disadvantage of requiring extra memory. There are two possible ways to overcome this drawback. The first is to convert (3.19) to a power series in F_N and use the Horner scheme to compute V_N^M. The disadvantage of this approach is its sensitivity to roundoff errors. The second is based on calculating the roots of $H_M(w)$. Let us assume that the roots are

$$\theta_1, \cdots, \theta_M. \tag{3.22}$$

Since the b_k are real, every complex root appears with its conjugate. Rearranging (3.22) in such a way that the first $2p$ roots are p conjugate pairs, we get

$$\mu_1, \bar{\mu}_1, \cdots, \mu_p, \bar{\mu}_p, \mu_{p+1}, \cdots, \mu_{M-p}. \tag{3.23}$$

Thus

$$H_M(w) = \alpha_0 \prod_{i=1}^{p} (1 - \alpha_i w + \beta_i w^2) \prod_{i=p+1}^{M-p} (1 - \gamma_i w), \tag{3.24}$$

while

$$\begin{aligned} &\alpha_0 = \sum_{k=0}^{M/2} b_k, \\ &\beta_i = 2R_e\mu_i/|\mu_i|^2, \quad \beta_i = 1/|\mu_i|^2, \quad 1 \leqq i \leqq p, \\ &\gamma_i = 1/\mu_i, \qquad p+1 \leqq i \leqq M-p. \end{aligned} \tag{3.25}$$

Hence we get

$$H_M(F_N) = \alpha_0 \prod_{i=1}^{p} [I - \alpha_i F_N + \beta_i F_N^2] \prod_{i=p+1}^{M-p} [I - \gamma_i F_N] U_N^0. \tag{3.26}$$

Each algorithm described above can be used as a one-step method by calculating the solution at the final time t directly from the initial data. It can also be used as a marching scheme when considering intermediate results. The size of the timestep Δt depends only on the information we want to obtain from the numerical procedure. Δt enters instead of t in the expressions above, and the parameter R is determined accordingly. In any case, the refinement of the algorithm is done by increasing the degree of the polynomial and not by decreasing the size of the timestep.

4. Accuracy and stability. Using (3.4), (3.15), and (3.17), we get

$$\|E\|_{L_2} \leqq 2\, e^{-R} \left| \sum_{k=M+1}^{\infty} I_k(R)\, T_k(w) \right|, \qquad -1 \leqq w \leqq 1. \tag{4.1}$$

Since e^{Rw} is an entire function, it satisfies the following theorem ([4, pp. 94-96]).

THEOREM (S.N. Bernstein). *Let $f(w)$ be an entire transcendental function that is real for real w. Then there exists a sequence of integers $n_1, n_2, \cdots$ with $n_\mu \to \infty$ such that the relation*

$$\lim_{\mu \to \infty} \frac{E_{n_\mu}(f)}{|\alpha_{n_\mu} + 1|} = \tag{4.2}$$

holds, where α_k are the coefficients in the expansion

$$f(w) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} \alpha_k T_k(w) \tag{4.3}$$

and

$$E_n(f)=\left|f(w)-\frac{\alpha_0}{2}-\sum_{k=1}^{n}\alpha_k T_k(w)\right|. \tag{4.4}$$

There is a sequence of integers n_μ, $\mu=1,2,\cdots$ *of the above type provided*

$$\alpha_{n_\mu+1}\neq 0, \qquad \mu=1,2,\cdots, \text{ and} \tag{4.5}$$

(1)

$$\sum_{k=n_\mu+2}^{\infty}|\alpha_k|=O(|\alpha_{n_\mu+1}|) \quad \text{as } \mu\to\infty. \tag{4.6}$$

(2)

In our case we can take $n_\mu=\mu$, $\mu=1,2,\cdots$, and it follows that

$$\|E\|_{L_2}\leqq 2\,e^{-R}I_{M+1}(R)(1+O(1)). \tag{4.7}$$

The asymptotic expansion of $I_k(R)$ is [1]:

$$I_k(R)\sim\frac{e^R}{\sqrt{2\pi R}}\left\{1-\frac{\mu-1}{8R}+\frac{(\mu-1)(\mu-9)}{2!(8R)^2}-\frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8R)^3}+\cdots\right\} \tag{4.8}$$

where

$$\mu=4k^2. \tag{4.9}$$

Hence,

$$2\,e^{-R}I_k(R)\sim\sqrt{\frac{2}{\pi R}}\left[1-\frac{\mu}{8R}+\frac{1}{2!}\left(\frac{\mu}{8R}\right)^2-\cdots+O\left(\frac{1}{R}\right)\right] \tag{4.10}$$

or

$$2\,e^{-R}I_k(R)\sim\sqrt{\frac{2}{\pi R}}\exp(-\mu/8R)+O(R^{-3/2}). \tag{4.11}$$

From (4.7), (4.9), and (4.11), we conclude that an ε time accuracy,

$$\|E\|_{L_2}\leqq\varepsilon, \tag{4.12}$$

is achieved when

$$M=O(R^{1/2}). \tag{4.13}$$

It is clear that satisfying (4.13) guarantees stability. In fact, using (3.1), (4.12), we get

$$\|\exp(tG_N)-H_M(tG_N)\|\leqq\varepsilon; \tag{4.14}$$

hence,

$$\|H_M(tG_N)\|\leqq\|\exp(tG_N)\|+\varepsilon. \tag{4.15}$$

Since $\exp(tG_N)$ is a stable operator [3], $H_M(tG_N)$ is stable as well.

R is equal to $aN^2t/2$; thus from (4.13) we can conclude the main result of this analysis. In order to achieve ε time accuracy, stable solution of (2.3), M must satisfy

$$M=O(N). \tag{4.16}$$

A similar analysis for any finite-order scheme based on Taylor expansion of e^z will imply that M [$M=O(1/\Delta t)$; see (3.7)-(3.8)] must be proportional to N^2; thus the advantage of the orthogonal polynomials approach is obvious.

ALGORITHM REFINEMENT. From (3.13), (4.7), (4.9), and (4.11), we get

$$E \approx \frac{2}{N}\left(\frac{1}{a\pi t}\right)^{1/2} \exp\left(-\frac{(M/N)^2}{at}\right). \tag{4.17}$$

Expression (4.17) suggests refinement of the algorithm while

$$M = N^{\alpha} \qquad (\alpha > 1) \tag{4.18}$$

will yield an exponential decay of the error. The accuracy thus achieved is the desired spectral accuracy.

5. Numerical results. Table 1 presents the stability properties of the OPS (Orthogonal Polynomial Scheme) compared to the modified Euler scheme, which is second order in time. We used the model problem (2.1) with $a = 1$, and initial data

$$U^0(x) = \sin(3x). \tag{5.1}$$

The solution is computed at $t = 1$. M indicates the minimal number of applications of the operator tG_N we must use to achieve stable (meaningful) results.

Table 2 clarifies the spectral convergence of the OPS scheme. In this table we included the results for the modified Euler scheme as well for the sake of comparison. The problem solved is

$$\begin{aligned} &U_t - U_{xx} = 0, \\ &U^0(x) = x(x-2\pi), \end{aligned} \qquad 0 \leqq x \leqq 2\pi. \tag{5.2}$$

Note that the periodic continuation of $U^0(x)$ belongs to C^0; thus the Fourier coefficients of $U^0(x)$ are decaying slowly. The solution is computed at $t = 1$. The refinement of the modified Euler scheme is done while M satisfies

$$M = 0.97 \times (N/2)^2.$$

For the OPS algorithm, M satisfies

$$M = 2.5 \times (N/2)^{1.2}.$$

TABLE 1

N	Modified Euler M	OPS M
16	48	24
32	192	48
64	768	96

TABLE 2

	Modified Euler			OPS		
N	M	L_2-error	Ratio	M	L_2-error	Ratio
16	62	.3791 − 04	17.4	26	.1026 − 04	92
32	250	.2126 − 05	16.2	61	.1107 − 06	134
64	1000	.1339 − 06		140	.8263 − 09	

The increasing ratio between the L_2-errors of two successive refinements verifies the spectral convergence of the OPS algorithm.

In Table 3 we compare the OPS to the modified Euler scheme from the point of view of the amount of work needed to achieve a certain degree of accuracy. The problem solved is $U_t - U_{xx} = 0$ with $U^0(x) = \sin(3x)$. The L_2-error is computed at the time level $t = 1$, and the space resolution is $N = 32$.

In the next set of numerical experiments we used our algorithm to solve a variable coefficient problem, which can be presented as follows:

$$\begin{aligned} &(u_N)_t - G_N u_N = s_1(x) + t s_2(x), \\ &u_N(x, 0) = u_N^0(x), \end{aligned} \tag{5.3}$$

while

$$G_N = P_N G P_N, \tag{5.4a}$$

$$G = a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x} + c(x). \tag{5.4b}$$

If the exact solution of (5.3) is

$$u_N(x, t) = t s(x), \tag{5.5}$$

then

$$u_N^0(x) = 0, \tag{5.6}$$

$$s_1(x) = s(x), \tag{5.7}$$

$$s_2(x) = -G_N(S_1(x)). \tag{5.8}$$

The formal solution of (5.3) is

$$u = f_0(G_N t)u_N^0 + f_1(G_N t)s_1 + f_2(G_N t)s_2 \tag{5.9}$$

where

$$f_0(G_N t) = \exp(G_N t), \tag{5.10}$$

$$f_1(G_N t) = \int_0^t \exp(G_N \tau)\, d\tau = t(G_N t)^{-1}(\exp(G_N t) - I), \tag{5.11}$$

$$f_2(G_N t) = \int_0^t \exp(G_N \tau)(t - \tau)\, d\tau = t^2(G_N t)^{-2}(\exp(G_N t) - G_N t - I). \tag{5.12}$$

Since $u_N^0 = 0$, the first term on the right-hand side of (5.9) is zero. Thus, in order to implement our algorithm we have to approximate the following functions:

$$f_1(zt) = \int_0^t e^{z\tau}\, d\tau = t\frac{e^{zt} - 1}{zt} = t\tilde{f}_1(zt), \qquad z \in D, \tag{5.13}$$

$$f_2(zt) = \int_0^t e^{zt}(t - \tau)\, d\tau = t^2\frac{e^{zt} - zt - 1}{(zt)^2} = t^2\tilde{f}_2(zt), \qquad z \in D \tag{5.14}$$

TABLE 3

L_2-error	M (modified Euler)	M (OPS)
1.3×10^{-2}	200	50
1.3×10^{-4}	2,000	60
1.3×10^{-6}	20,000	70

where D is the domain in the complex plane that contains the eigenvalues of the operator G_N. The domain D can be approximated by doing a Fourier analysis of the constant coefficients operator

$$\tilde{G} = a\frac{\partial^2}{\partial x^2} + b\frac{\partial}{\partial x} + c \tag{5.15}$$

where

$$a = \max |a(x)|, \tag{5.16a}$$

$$b = \max |b(x)|, \tag{5.16b}$$

$$c_{\max} = \max |c(x)|; \qquad c_{\min} = \min |c(x)|. \tag{5.16c}$$

We get

$$\tilde{D} = \{x + iy \,|\, -(N^2 + c_{\max}) \leqq x \leqq -c_{\min}, \, |y| \leqq b\}. \tag{5.17}$$

Since

$$N^2 + C_{\max} \gg b, \tag{5.18}$$

using the domain $\tilde{\tilde{D}}$ where

$$\tilde{\tilde{D}} = \{x \,|\, -(N^2 + c_{\max}) \leqq x \leqq -c_{\min}\}, \tag{5.19}$$

we obtain a good approximation of D. In order to write $\tilde{f}_1, \tilde{f}_2$ as a Chebyshev polynomial expansion we first must change variables. Define

$$z = Rw + Q, \qquad -1 \leqq w \leqq 1 \tag{5.20}$$

where

$$R = \tfrac{1}{2}[N^2 + c_{\max} - c_{\min}], \tag{5.21}$$

$$Q = -\tfrac{1}{2}[N^2 + c_{\max} + c_{\min}]; \tag{5.22}$$

then

$$\tilde{f}_1(zt) = \hat{f}_1(wt) = \frac{e^{Qt}\, e^{Rtw} - 1}{Rtw + Qt} = \sum_{k=0}^{\infty} d_k T_k(w), \tag{5.23}$$

$$\tilde{f}_2(zt) = \frac{1}{zt}[\tilde{f}_1(zt) - 1] = \frac{\hat{f}_1(wt) - 1}{Rtw + Qt} = \sum_{k=0}^{\infty} g_k T_k(w). \tag{5.24}$$

From (3.14), (3.15), (3.16) we have

$$e^{Rtw} = \sum_{k=0}^{\infty} b_k T_k(w), \tag{5.25}$$

while

$$b_k = c_k I_k(Rt), \qquad c_k = \begin{cases} 1, & k = 1, \\ 2, & k \geqq 2, \end{cases} \tag{5.26}$$

and I_k are modified Bessel functions. Substituting (5.25) in (5.13) and (5.24), we get

$$Rw \sum_{k=0}^{\infty} d_k T_k + Q \sum_{k=0}^{\infty} d_k T_k = \frac{1}{t}\left[\sum_{k=0}^{\infty} \tilde{b}_k T_k - 1\right] \qquad (\tilde{b}_k = e^{Qt} b_k), \tag{5.27}$$

$$Rw \sum_{k=0}^{\infty} g_k T_k + Q \sum_{k=0}^{\infty} g_k T_k = \frac{1}{t}\left[\sum_{k=0}^{\infty} d_k T_k - 1\right]. \tag{5.28}$$

Since Chebyshev polynomials satisfy the recurrence relation

$$T_{k+1} = 2wT_k - T_{k-1}, \tag{5.29}$$

we get

$$wT_k = \tfrac{1}{2}(T_{k+1} + T_{k-1}). \tag{5.30}$$

Using (5.30) in (5.27), (5.28) results in tridiagonal systems of equations for the two sets of unknowns:

$$\{d_k\}, \quad \{g_k\}, \quad 0 \leqq k \leqq N/2.$$

In our numerical experiments, we used

$$s_1(x) = \exp(-4(x-\pi)^2), \tag{5.31}$$

$$a(x) = 1./(2.+\cos x), \tag{5.32}$$

$$b(x) = 1./(2.+\sin x), \tag{5.33}$$

$$c(x) = -20./(2.+\cos x). \tag{5.34}$$

The results at $t = 1$ are presented in Table 4.

We also used the OPS algorithm to compute the solution at $t = 10$, and the results are presented in Table 5.

Observing Tables 4 and 5, we notice that M does not depend on t. This can be explained as follows. For large t, $\|\exp(G_N t)\|$ is very small. Thus, since (5.11), (5.12), we have

$$\tilde{f}_1(G_N t) \approx -G_N^{-1}, \tag{5.35}$$

$$\tilde{f}_1(G_N t) \approx -G_N^{-2}(G_N t + I), \tag{5.36}$$

which means that for large t, approximating $\tilde{f}_1(G_N t)$ is equivalent to inverting G_N and approximating $\tilde{f}_2(G_N t)$ is equivalent to inverting $(G_N)^2$.

6. Conclusion. The algorithm presented in this paper achieves the goal of spectral accuracy in time and space for the simple model problem (2.1). We believe that this approach can be useful for more complicated problems. In fact, the scheme described in § 3 is applicable whenever we can represent the solution as $\exp(tG_N)U_N$ and the eigenvalues of tG_N are grouped close to the real axis.

TABLE 4

Modified Euler			OPS	
N	M	L_2-error	M	L_2-error
16	30	.4885−01	16	.4885−01
32	120	.1250−05	40	.1173−05
64	480	.1296−05	100	.1258−08

TABLE 5

N	M	L_2-error (OPS)
16	16	.5358−01
32	40	.2377−05
64	110	.3968−07

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of Mathematical Functions*, Dover, New York, 1972.

[2] W. J. CODY, G. MEINARDUS, AND R. S. VARGA, *Chebyshev, rational approximations to heat—conduction problems*, J. Approx. Theory, 2 (1969), pp. 50–65.

[3] D. GOTTLIEB AND S. ORSZAG, *Numerical Analysis of Spectral Methods: Theory and Applications*, CBMS–NSF Regional Conference Series in Applied Mathematics 26, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1972.

[4] G. MEIMARDUS, *Approximation of Functions: Theory and Numerical Methods.* Springer-Verlag, Berlin, New York, 1967.

[5] H. TAL-EZER, *Spectral methods in time for hyperbolic problems*, SIAM J. Numer. Anal., 23 (1986), pp. 11–26.

[6] R. S. VARGA, *Functional Analysis and Approximation Theory in Numerical Analysis*, CBMS–NSF Regional Conference Series in Applied Mathematics 3, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1971.

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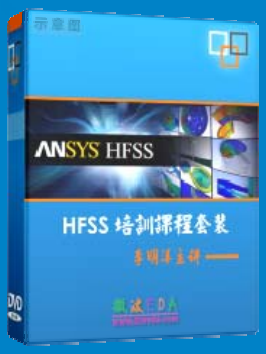

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