Appendix A
Fourier Transform

Appendix A

- **Appendix A Fourier Transform**
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A.1 Fourier Series

☞ [Fourier Series] **Let the signal *x(t)* be a periodic signal with period T_0.If the following conditions are satisfied**

1. *x(t)* is absolutely integrable over its period

$$\int_0^{T_0} |x(t)|dt < \infty$$

2. The number of maxim and minima of x(t) in each period is finite

3. The number of discontinuous of x(t) in each period is finite

then x(t) can be expanded in terms of the complex exponential signal as

$$x_{\pm}(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi\frac{n}{T_0}t}$$ **where** $$x_n = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{-j2\pi\frac{n}{T_0}t}dt$$

for some arbitrary α **and**

$$x_{\pm}(t) = \begin{cases} x(t) & \text{if x(t) is continous at t} \\ (x(t^+)+x(t^-))/2 & \text{if x(t) is discontinous at t} \end{cases}$$

A.1 Fourier Series

- x_n are called the Fourier series coefficients of the signal *x(t)*.
- For all practice purpose, $x_{\pm}(t) = x(t)$
- From now on, we will use $x(t)$ instead of $x_{\pm}(t)$
- The quantity $f_0 = 1/T_0$ is called the fundamental frequency of the signal *x(t)*
- The Fourier series expansion can be expressed in terms of angular frequency $\omega_0 = 2\pi f_0$ by

$$x_n = \frac{\omega_0}{2\pi}\int_{\alpha}^{\alpha+2\pi/\omega_0} x(t)e^{-jn\omega_0 t}dt$$

and

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t}$$

A.1 Fourier Series

- **Discrete spectrum - we may represent $x_n = |x_n| e^{j\angle x_n}$ where $|x_n|$ gives the magnitude of the *n*th harmonic and $\angle x_n$ gives its pha**

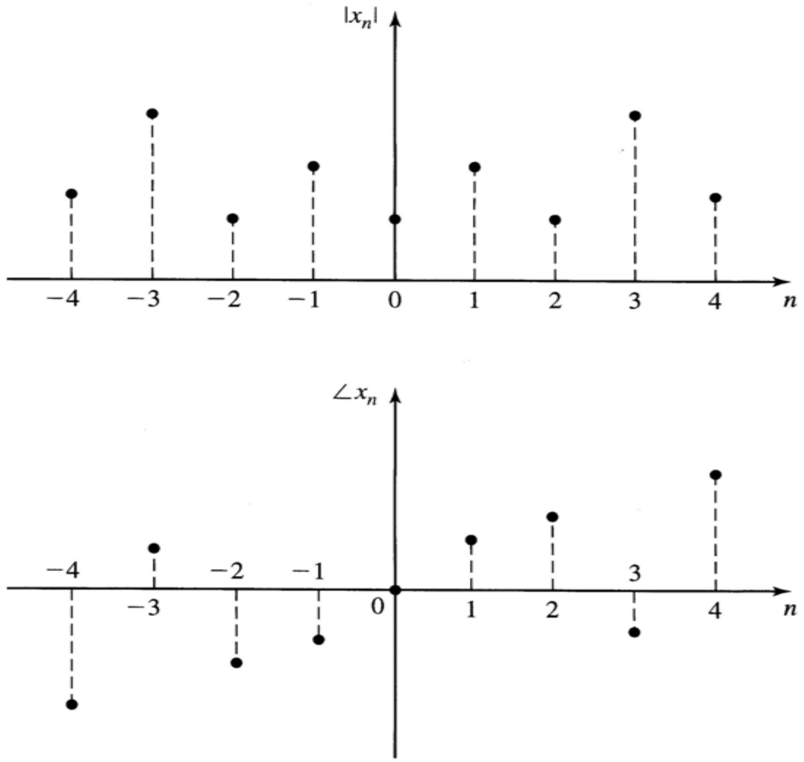


Figure 2.1 The discrete spectrum of $x(t)$.

A.1 Fourier Series

- **Example: Let *x*(t) denote the periodic signal depicted in Figure 2.2**

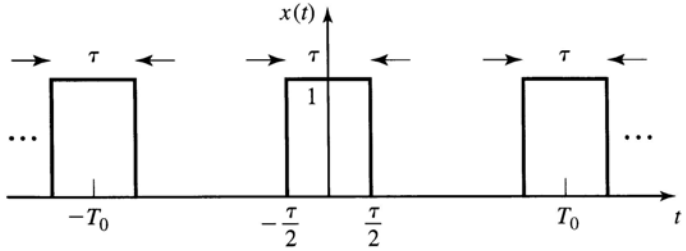


Figure 2.2 Periodic signal $x(t)$.

$$x(t) = \sum_{n=-\infty}^{\infty} \Pi\left(\frac{t - nT_0}{\tau}\right)$$

where

$$\Pi(t) = \begin{cases} 1 & |t| < 1/2 \\ 1/2 & |t| = 1/2 \\ 0 & otherwise \end{cases}$$

is a rectangular pulse. Determine the Fourier series expansion for this signal

A.1 Fourier Series

Solution: We first observe that the period of the signal is T_0 and

$$x_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t) e^{-jn\frac{2\pi t}{T_0}} dt$$

$$= \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} 1 e^{-jn\frac{2\pi t}{T_0}} dt$$

$$= \frac{1}{T_0}\frac{T_0}{-jn2\pi}\left[e^{-jn\frac{n\tau}{T_0}} - e^{jn\frac{n\tau}{T_0}}\right]$$

$$= \frac{1}{\pi n}\sin\left(\frac{n\pi\tau}{T_0}\right)$$

$$= \frac{\tau}{T_0}\text{sinc}\left(\frac{n\tau}{T_0}\right)$$

Therefore, we have

$$x(t) = \sum^{\infty} \frac{\tau}{T_0} \operatorname{sinc}\left(\frac{n\tau}{T_0}\right) e^{jn\frac{2\pi t}{T_0}}$$

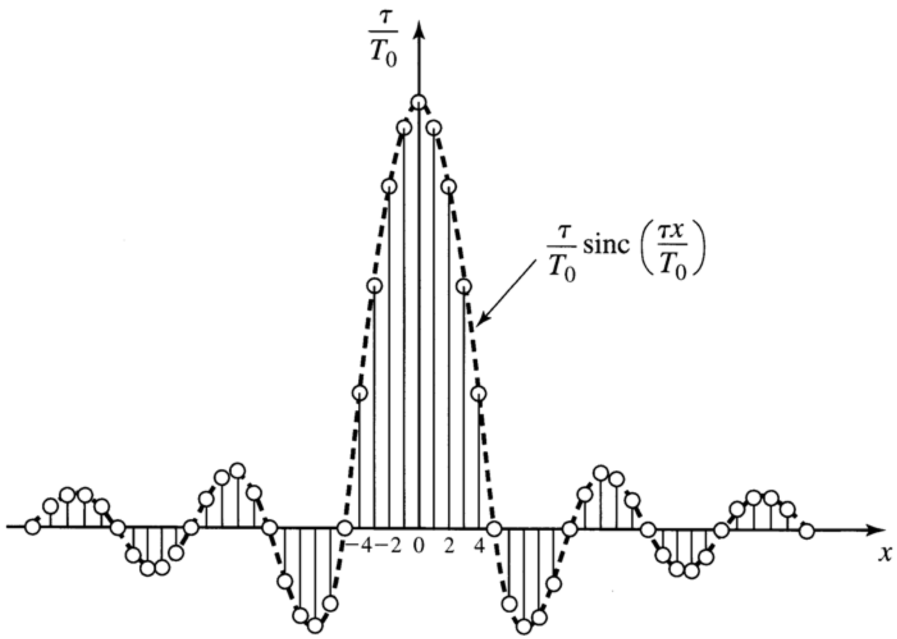
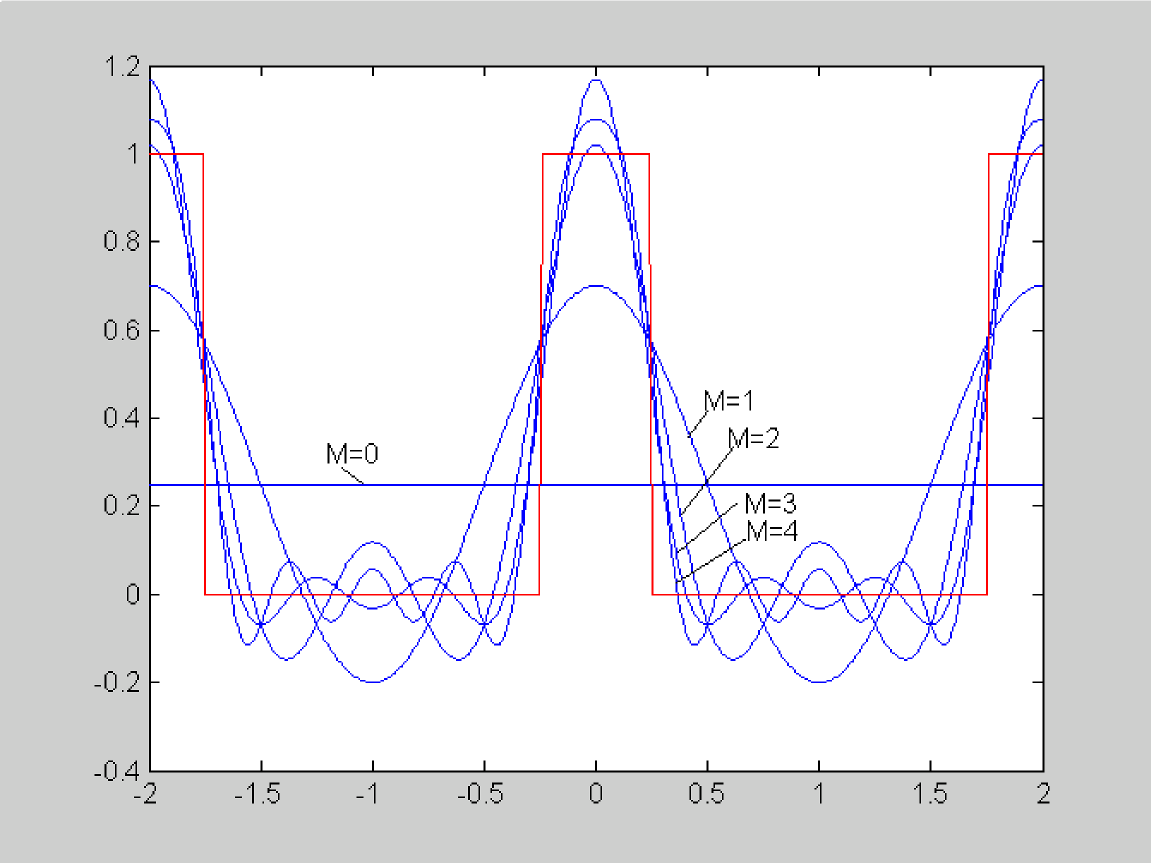


Figure 2.3 The discrete spectrum of the rectangular pulse train.

A.1 Fourier Series

☞ **Superposition of** $x(t) = \sum_{n=-M}^{M} \frac{\tau}{T_0} \mathrm{sinc}\left(\frac{n\tau}{T_0}\right) e^{jn\frac{2\pi t}{T_0}}$

$\tau = 0.5 \qquad T = 2$



A.1 Fourier Series for Real Signals

☞ **If the signal *x(t)* is a real signal, we have**

$$x_{-n} = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{j2\pi\frac{n}{T_0}t}\,dt$$

$$= \left[\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{-j2\pi\frac{n}{T_0}t}\,dt\right]^*$$

$$= x_n^*$$

A.1 Fourier Series for Real Signals

- **For a real, periodic x(t), the positive and negative coefficients are conjugates.**
- $|x_n|$ **has even symmetry and** $\angle x_n$ **has odd symmetry with respect to the** ***n=0*** **axis.**

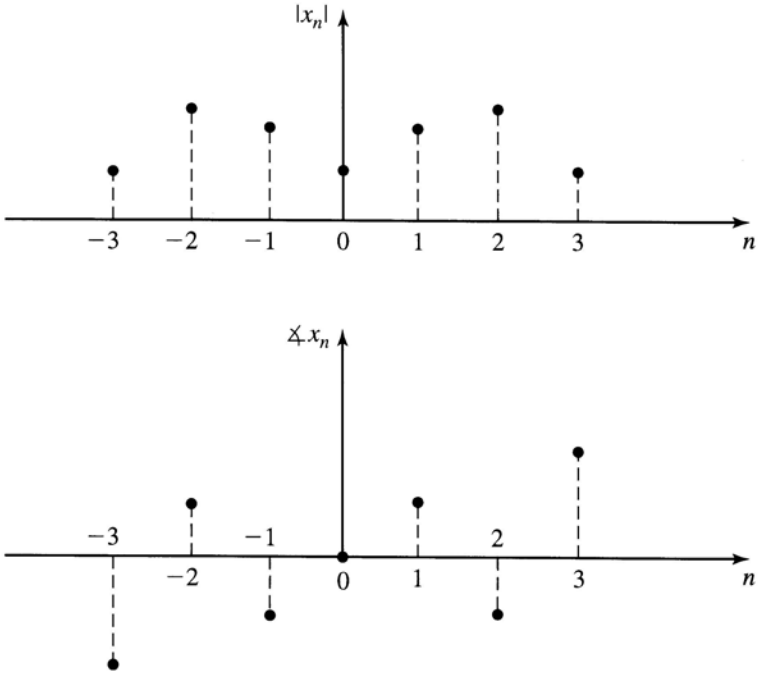


Figure 2.5 Discrete spectrum of a real-valued signal.

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A.1 Fourier Series for Real Signals

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi \frac{n}{T_0}t} \qquad x_n = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{-j2\pi\frac{n}{T_0}t}\,dt$$

- **We may let** $x_n = \dfrac{a_n - jb_n}{2}$ **and** $x_{-n} = \dfrac{a_n + jb_n}{2}$

$$x_n e^{j2\pi\frac{n}{T_0}t} + x_{-n} e^{-j2\pi\frac{n}{T_0}t} = \frac{a_n - jb_n}{2} e^{j2\pi\frac{n}{T_0}t} + \frac{a_n + jb_n}{2} e^{-j2\pi\frac{n}{T_0}t}$$

$$= a_n \cos\left(2\pi \frac{n}{T_0} t\right) + b_n \sin\left(2\pi \frac{n}{T_0} t\right)$$

- **Since** x_0 **is real and** $x_0 = \dfrac{a_0}{2}$ **, we conclude that**

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(2\pi \frac{n}{T_0} t\right) + b_n \sin\left(2\pi \frac{n}{T_0} t\right)\right]$$

- **This relation is called the trigonometric Fourier series expansion.**

A.1 Fourier Series for Real Signals

☞ **To obtain** a_n **and** b_n **, we have**

$$x_n = \frac{a_n - jb_n}{2} = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t) e^{-j2\pi\frac{n}{T_0}t} dt$$

$$= \underbrace{\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)\cos\left(2\pi\frac{n}{T_0}t\right)dt}_{\frac{a_n}{2}} - \underbrace{\frac{j}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)\sin\left(2\pi\frac{n}{T_0}t\right)dt}_{\frac{jb_n}{2}}$$

☞ **From above equation, we obtain**

$$a_n = \frac{2}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)\cos\left(2\pi\frac{n}{T_0}t\right)dt$$

$$b_n = \frac{2}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)\sin\left(2\pi\frac{n}{T_0}t\right)dt$$

A.1 Fourier Series for Real Signals

- **There exists a third way to represent the Fourier series expansion of a real signal. Noting that**

$$x_n e^{j2\pi \frac{n}{T_0}t} + x_{-n} e^{-j2\pi \frac{n}{T_0}t} = 2|x_n|\cos\left(2\pi \frac{n}{T_0}t + \angle x_n\right)$$

we have

$$x(t) = x_0 + 2\sum_{n=1}^{\infty} |x_n| \cos\left(2\pi \frac{n}{T_0}t + \angle x_n\right)$$

- **For a real periodic signal, we have three alternative ways to represent Fourier series expansion**

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} x_n e^{j2\pi \frac{n}{T_0}t} \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(2\pi \frac{n}{T_0}t\right) + b_n \sin\left(2\pi \frac{n}{T_0}t\right)\right] \\ &= x_0 + 2\sum_{n=1}^{\infty} |x_n| \cos\left(2\pi \frac{n}{T_0}t + \angle x_n\right) \end{aligned}$$

A.1 Fourier Series for Real Signals

☞ **The corresponding coefficients are obtained from**

$$x_n = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{-j2\pi\frac{n}{T_0}t}dt = \frac{a_n}{2} + j\frac{b_n}{2}$$

$$a_n = \frac{2}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)\cos\left(2\pi\frac{n}{T_0}t\right)dt$$

$$b_n = \frac{2}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)\sin\left(2\pi\frac{n}{T_0}t\right)dt$$

$$|x_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$$

$$\angle x_n = -\arctan\left(\frac{b_n}{a_n}\right)$$

A.1 Fourier Series

Complex Exponential Fourier Series

- **Given a signal $x(t)$ defined over the interval (t_0, t_0+T_0) with the definition**

$$w_0 = 2 \cdot \pi \cdot f_0 = \frac{2\pi}{T_0}$$

we define the complex exponential Fourier series as

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jnw_0 t} \quad , t_0 \le t \le t_0 + T_0$$

where

$$X_n = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jnw_0 t} dt$$

EXAMPLE 2.6 Consider the signal

$$x(t) = \cos \omega_0 t + \sin^2 2\omega_0 t \tag{2.45}$$

where $\omega_0 = 2\pi/T_0$. Find the complex exponential Fourier series.

SOLUTION We could compute the Fourier coefficients using (2.44), but by using appropriate trigonometric identities and Euler's theorem, we obtain

$$\begin{aligned} x(t) &= \cos \omega_0 t + \tfrac{1}{2} - \tfrac{1}{2}\cos 4\omega_0 t \\ &= \tfrac{1}{2}e^{j\omega_0 t} + \tfrac{1}{2}e^{-j\omega_0 t} + \tfrac{1}{2} - \tfrac{1}{4}e^{j4\omega_0 t} - \tfrac{1}{4}e^{-j4\omega_0 t} \end{aligned} \tag{2.46}$$

Invoking uniqueness and equating the second line term by term with $\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$, we find that

$$\begin{aligned} X_0 &= \tfrac{1}{2} \\ X_1 &= \tfrac{1}{2} = X_{-1} \\ X_4 &= -\tfrac{1}{4} = X_{-4} \end{aligned} \tag{2.47}$$

with all other X_n's equal to zero. Thus considerable labor is saved by noting that the Fourier series of a signal is unique.

A.1 Fourier Series

Trigonometric form of the Fourier series

- **If x(t) is real and periodic signal**

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jnw_0 t} \quad , t_0 \le t \le t_0 + T_0$$

$$\Rightarrow x(t) = X_0 + \sum_{n=1}^{\infty} 2\left|X_n\right| \cos\left(nw_0 t + \angle X_n\right)$$

$$\Rightarrow x(t) = X_0 + \sum_{n=1}^{\infty} A_n \cos nw_0 t + \sum_{n=1}^{\infty} B_n \sin nw_0 t$$

where

$$A_n = 2\left|X_n\right| \cos\left(\angle X_n\right) = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} x(t) \cos\left(nw_0 t\right) dt$$

$$B_n = -2\left|X_n\right| \sin\left(\angle X_n\right) = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} x(t) \sin\left(nw_0 t\right) dt$$

☞ Parseval's Theorem

$$P = \frac{1}{T_0}\int_{T_0} \left|x(t)\right|^2 dt = \sum_{n=-\infty}^{\infty} \left|x(t)\right|^2$$

$$= X_0^2 + 2\sum_{n=1}^{\infty} \left|x(t)\right|^2$$

A.1 Fourier Series

☞ Fourier series for several periodic signals

TABLE 2.1 Fourier Series for Several Periodic Signals

Signal: $-\frac{1}{2}T_0 \leq t \leq \frac{1}{2}T_0$ (One Period)	Coefficients for Exponential Fourier Series
1. Asymmetrical pulse train: $x(t) = A\Pi\left(\frac{t-t_0}{\tau}\right),\ \tau < T_0$; $x(t) = x(t+T_0)$, all t	$X_n = \frac{A\tau}{T_0}\operatorname{sinc}(nf_0\tau)e^{-j2\pi nf_0t_0}$; $n = 0, \pm 1, \pm 2, \ldots$
2. Half-rectified sinewave: $x(t) = \begin{cases} A\sin\omega_0 t, & 0 \leq t \leq \frac{1}{2}T_0 \\ 0, & -\frac{1}{2}T_0 \leq t \leq 0 \end{cases}$; $x(t) = x(t+T_0)$, all t	$X_n = \begin{cases} \frac{A}{\pi(1-n^2)}, & n = 0, \pm 2, \pm 4, \ldots \\ 0, & n = \pm 3, \pm 5, \ldots \\ -\frac{1}{4}jA, & n = 1 \\ \frac{1}{4}jA, & n = -1 \end{cases}$
3. Full-rectified sinewave: $x(t) = A\lvert\sin\omega_0 t\rvert$	$X_n = \begin{cases} \frac{2A}{\pi(1-n^2)}, & n = 0, \pm 2, \pm 4, \ldots \\ 0, & n = \pm 1, \pm 3, \ldots \end{cases}$
4. Triangular wave: $x(t) = \begin{cases} \frac{4A}{T_0}t + A, & -\frac{1}{2}T_0 \leq t < 0 \\ -\frac{4A}{T_0}t + A, & 0 \leq t \leq \frac{1}{2}T_0 \end{cases}$; $x(t) = x(t+T_0)$, all t	$X_n = \begin{cases} \frac{4A}{\pi^2 n^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$

A.2 Fourier transform

- **Fourier transform is the extension of Fourier series to periodic and nonperiodic signals.**
- **The signal are expressed in terms of complex exponentials of various frequencies, but these frequencies are not discrete.**
- **The signal has a continuous spectrum as opposed to a discrete spectrum.**

A.2 Fourier transform

☞ Theorem 2.2.1 [Fourier Transform] **If the signal *x(t)* satisfies certain conditions known as Dirichlet conditions, namely,**

1. *x(t)* is absolutely integrable on the real line, i.e.,

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

2. The number of maxima and minima of x(t) in any finite interval on the real line is finite,

3. The number of discontinuities of x(t) in any finite interval on the real line is finite,

Then, the Fourier transform of x(t), defined by

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

And the original signal can be obtained from its Fourier transform by

$$x_{\pm}(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

A.2 Fourier transform

- **Observations**
  - ***X(f)* is in general a complex function. The function *X(f)* is sometimes referred to as the *spectrum* of the signal *x(t)*.**
  - **To denote that *X(f)* is the Fourier transform of *x(t)*, the following notation is frequently employed**

$$X(f) = F[x(t)]$$

to denote that x(t) is the inverse Fourier transform of X(f) , the following notation is used

$$x(t) = F^{-1}[X(f)]$$

Sometimes the following notation is used as a shorthand for both relations

$$x(t) \Leftrightarrow X(f)$$

A.2 Fourier transform

- The Fourier transform and the inverse Fourier transform relations can be written as

$$x(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} d\tau \right] e^{j2\pi ft} df$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{j2\pi f(t-\tau)} df \right] x(\tau) d\tau$$

On the other hand,

$$x(t) = \int_{-\infty}^{\infty} \delta(t-\tau) x(\tau) d\tau$$

where $\delta(t)$ is the unit impulse. From above equation, we may have

$$\delta(t-\tau) = \int_{-\infty}^{\infty} e^{j2\pi f(t-\tau)} df$$

or, in general

$$\delta(t) = \int_{-\infty}^{\infty} e^{j2\pi ft} df$$

hence, the spectrum of $\delta(t)$ is equal to unity over all frequencies.

A.2 Fourier transform

Example 2.2.1: Determine the Fourier transform of the signal $\Pi(t)$.

Solution: We have

$$F[\Pi(t)] = \int_{-\infty}^{\infty} \Pi(t) e^{-j2\pi ft} dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \Pi(t) e^{-j2\pi ft} dt$$

$$= \frac{1}{-j2\pi f} \left[e^{-j\pi f} - e^{j\pi f} \right]$$

$$= \frac{\sin(\pi f)}{\pi f}$$

$$= \mathrm{sinc}(f)$$

A.2 Fourier transform

•The Fourier transform of $\Pi(t)$.

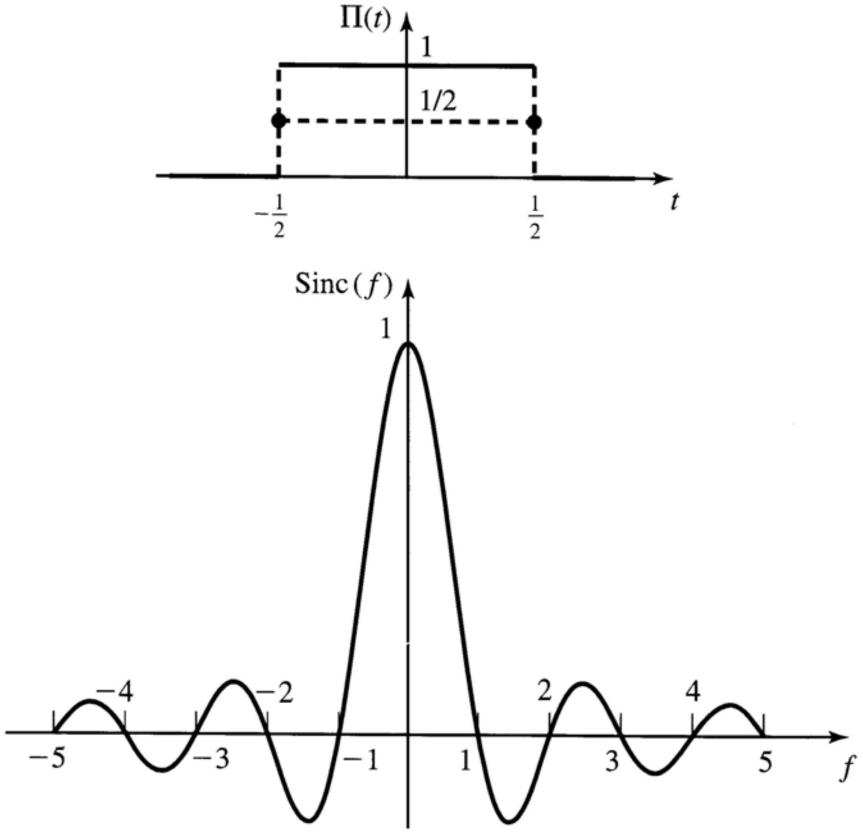


Figure 2.6 $\Pi(t)$ and its Fourier transform.

A.2 Fourier transform

Example 2.2.2: Find the Fourier transform of the impulse signal $x(t) = \delta(t)$ **.**

Solution: The Fourier transform can be obtained by

$$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt$$

$$= 1$$

Similarly, from the relation

$$\int_{-\infty}^{\infty} \delta(t) e^{j2\pi ft} dt = 1$$

We conclude that $F[1] = \delta(f)$

A.2 Fourier transform

•The Fourier transform of $\delta(t)$.

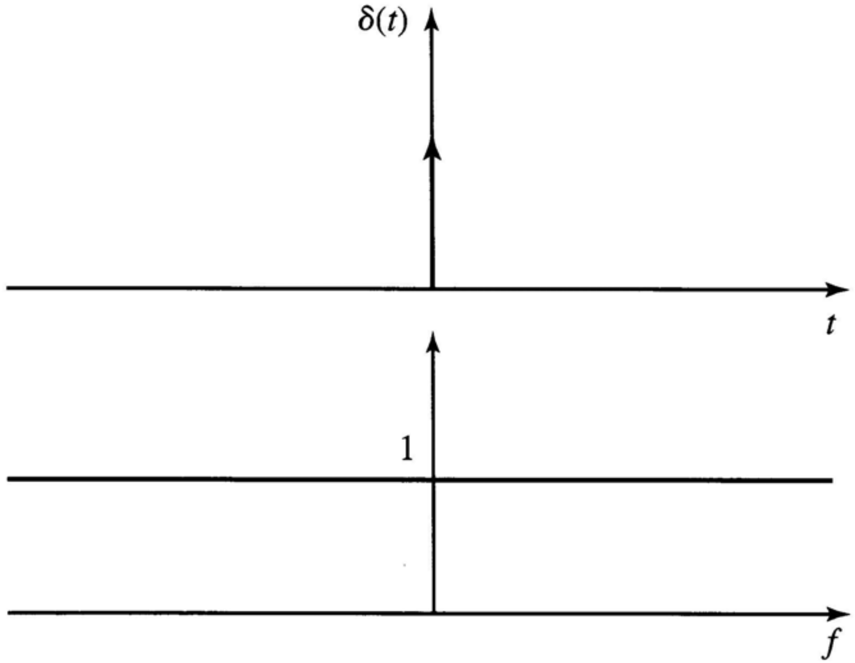


Figure 2.7 Impulse signal and its spectrum.

A.2 Fourier Transform of Real, Even, and Odd Signals

☞ **The Fourier transform can be written in general as**

$$F[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} x(t)\cos(2\pi ft)dt - j\int_{-\infty}^{\infty} x(t)\sin(2\pi ft)dt$$

For real x(t),

$$\int_{-\infty}^{\infty} x(t)\cos(2\pi ft)dt \text{ is real}$$

$$\int_{-\infty}^{\infty} x(t)\sin(2\pi ft)dt \text{ is real}$$

A.2 Fourier Transform of Real, Even, and Odd Signals

•Since cosine is an even function and sine is an odd function, we see that, for real *x(t)*, the real part of *X(f)* is an even function of *f* and the imaginary part is an odd function of *f*. Therefore, we have

$$X(-f) = X^*(f)$$

•This is equivalent to the following relations:

$$\mathrm{Re}[X(-f)] = \mathrm{Re}[X(f)]$$
$$\mathrm{Im}[X(-f)] = -\mathrm{Im}[X(f)]$$
$$|X(-f)| = |X(f)|$$
$$\angle X(-f) = -\angle X(f)$$

TABLE 2.1 TABLE OF FOURIER TRANSFORMS

Time Domain ($x(t)$)	Frequency Domain ($X(f)$)
$\delta(t)$	1
1	$\delta(f)$
$\delta(t-t_0)$	$e^{-j2\pi f t_0}$
$e^{j2\pi f_0 t}$	$\delta(f-f_0)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)$
$\sin(2\pi f_0 t)$	$-\frac{1}{2j}\delta(f+f_0)+\frac{1}{2j}\delta(f-f_0)$
$\Pi(t)=\begin{cases}1, & \|t\|<\frac{1}{2}\\ \frac{1}{2}, & t=\pm\frac{1}{2}\\ 0, & \text{otherwise}\end{cases}$	$\text{sinc}(f)$
$\text{sinc}(t)$	$\Pi(f)$
$\Lambda(t)=\begin{cases}t+1, & -1\le t<0\\ -t+1, & 0\le t<1\\ 0, & \text{otherwise}\end{cases}$	$\text{sinc}^2(f)$
$\text{sinc}^2(t)$	$\Lambda(f)$
$e^{-\alpha t}u_{-1}(t), \alpha>0$	$\frac{1}{\alpha+j2\pi f}$
$te^{-\alpha t}u_{-1}(t), \alpha>0$	$\frac{1}{(\alpha+j2\pi f)^2}$
$e^{-\alpha\|t\|}$	$\frac{2\alpha}{\alpha^2+(2\pi f)^2}$
$e^{-\pi t^2}$	$e^{-\pi f^2}$
$\text{sgn}(t)=\begin{cases}1, & t>0\\ -1, & t<0\\ 0, & t=0\end{cases}$	$1/(j\pi f)$
$u_{-1}(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\delta'(t)$	$j2\pi f$
$\delta^{(n)}(t)$	$(j2\pi f)^n$
$\frac{1}{t}$	$-j\pi\,\text{sgn}(f)$
$\sum_{n=-\infty}^{n=+\infty}\delta(t-nT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{n=+\infty}\delta\left(f-\frac{n}{T_0}\right)$

A.2 The Fourier Transform

☞ Properties

- **superposition theorem**

$$a_1 x_1(t) + a_2 x_2(t) \leftrightarrow a_1 X_1(f) + a_2 X_2(f)$$

- **Time-delay theorem**

$$x(t - t_0) \leftrightarrow X(f) e^{-j2\pi f t_0}$$

- **Scale-change theorem**

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

- **Duality theorem**

$$X(t) \leftrightarrow x(-f)$$

A.2 The Fourier Transform

- **Frequency translation theorem**

$$x(t)e^{j2\pi f_0 t} \leftrightarrow X(f - f_0)$$

- **Modulation theorem**

$$x(t)\cos(2\pi f_0 t) \leftrightarrow \frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$$

- **Differentiation theorem**

$$\frac{d^n x(t)}{dt^n} \leftrightarrow (j2\pi f)^n X(f)$$

- **Integration theorem**

$$\int_{-\infty}^{\lambda} x(\lambda)d\lambda \leftrightarrow (j2\pi f)^{-1} X(f) + \frac{1}{2}X(0)\delta(f)$$

A.2 The Fourier Transform

- **Convolution theorem**

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda)\, x_2(t-\lambda)\, dt = \int_{-\infty}^{\infty} x_1(t-\lambda)\, x_2(\lambda)\, dt \leftrightarrow X_1(f)\, X_2(f)$$

- **Multiplication theorem**

$$x_1(t) \cdot x_2(t) \leftrightarrow X_1(f) * X_2(f) = \int_{-\infty}^{\infty} X_1(\lambda)\, X_2(f-\lambda)\, df = \int_{-\infty}^{\infty} X_1(f-\lambda)\, X_2(\lambda)\, df$$

A.3 Discrete Time Fourier Transform 1/2

- **DTFT**

$$X(\omega) = \sum_{n--\infty}^{\infty} x(n) e^{-j\omega n}$$

- **IDTFT**

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

- **Parseval relations**

$$E_x = \sum_{n=-\infty}^{\infty} \left| x(n) \right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| X(\omega) \right|^2 d\omega$$

A.3 Discrete Time Fourier Transform 2/2

- **If $x(n)$ is absolutely summable**

  ➔ $X(w)$ exist

- **Low frequency : 0**

  High frequency : $\pm\pi$

- **Discrete time sequence**

  Continuous spectrum with periodic 2π

A.4 Discrete Fourier Transform

- ***N*-point DFT**

$$X(k)=\sum_{n=0}^{N-1}x(n)e^{-j\frac{2\pi nk}{N}}$$

- ***N*-point IDFT**

$$x(n)=\sum_{k=0}^{N-1}X(k)e^{j\frac{2\pi nk}{N}}$$

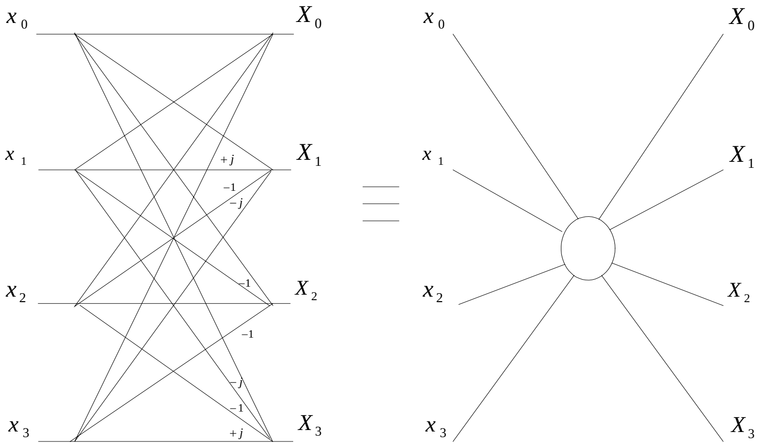
- **Discrete sequence**

  Discrete spectrum

- **Suitable for digital computer**

A.5 Fast Fourier Transform 1/5

- **Saving the computing and complex of DFT**
  - **The FFT drastically reduces the amount of calculations by exploiting the regularity of the operations in the DFT.**
- **Butterfly FFT**
  - **The radix-4 butterfly**
    - **In the radix-4 algorithm, the transform is split into a number of trivial four-point transforms, and non-trivial multiplications only have to be performed between stage of these four-point transforms.**



- **Suitable for digital computer**

CCU
Wireless Comm. Lab.

A.5 Fast Fourier Transform 2/5

- ☞ **An *N*-point DFT requires N^2 complex multiplications or phase rotations and N^2 complex additions.**

- ☞ **An *N*-point FFT using the radix-4 algorithm required only $\left(\frac{3}{8}\right)N\left(\log_2 N-2\right)$ complex multiplications or phase rotations and $N\log_2 N$ complex additions**

A.5 Fast Fourier Transform 3/5

☞ **FFT butterflies for an *N*-point DFT**

$$X_1(k) = \sum_{n=0}^{N-1} x_n W_N^{kn}$$

$$= \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_{2n} W_N^{2nk}}_{even\ sequence} + \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_{2n+1} W_N^{(2n+1)k}}_{odd\ sequence}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_{2n} W_N^{2nk} + W_N^k \sum_{n=0}^{\frac{N}{2}-1} x_{2n+1} W_N^{2nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_{2n} W_{\frac{N}{2}}^{nk} + W_N^k \sum_{n=0}^{\frac{N}{2}-1} x_{2n+1} W_{\frac{N}{2}}^{nk}$$

$$= X_{11}(k) + W_N^k X_{12}(k) \quad , \ k = 0,1,\ldots,N-1 \ .$$

$$W_N \triangleq e^{-j\frac{2\pi}{N}}$$

$$X_{11}(k) = X_{21}(k) + W_{N/2}^k X_{22}(k),$$

$$X_{12}(k) = X_{23}(k) + W_{N/2}^k X_{24}(k),$$

$$X_{21}(k) = x_0 + W_{N/4}^k x_4 \ ,$$

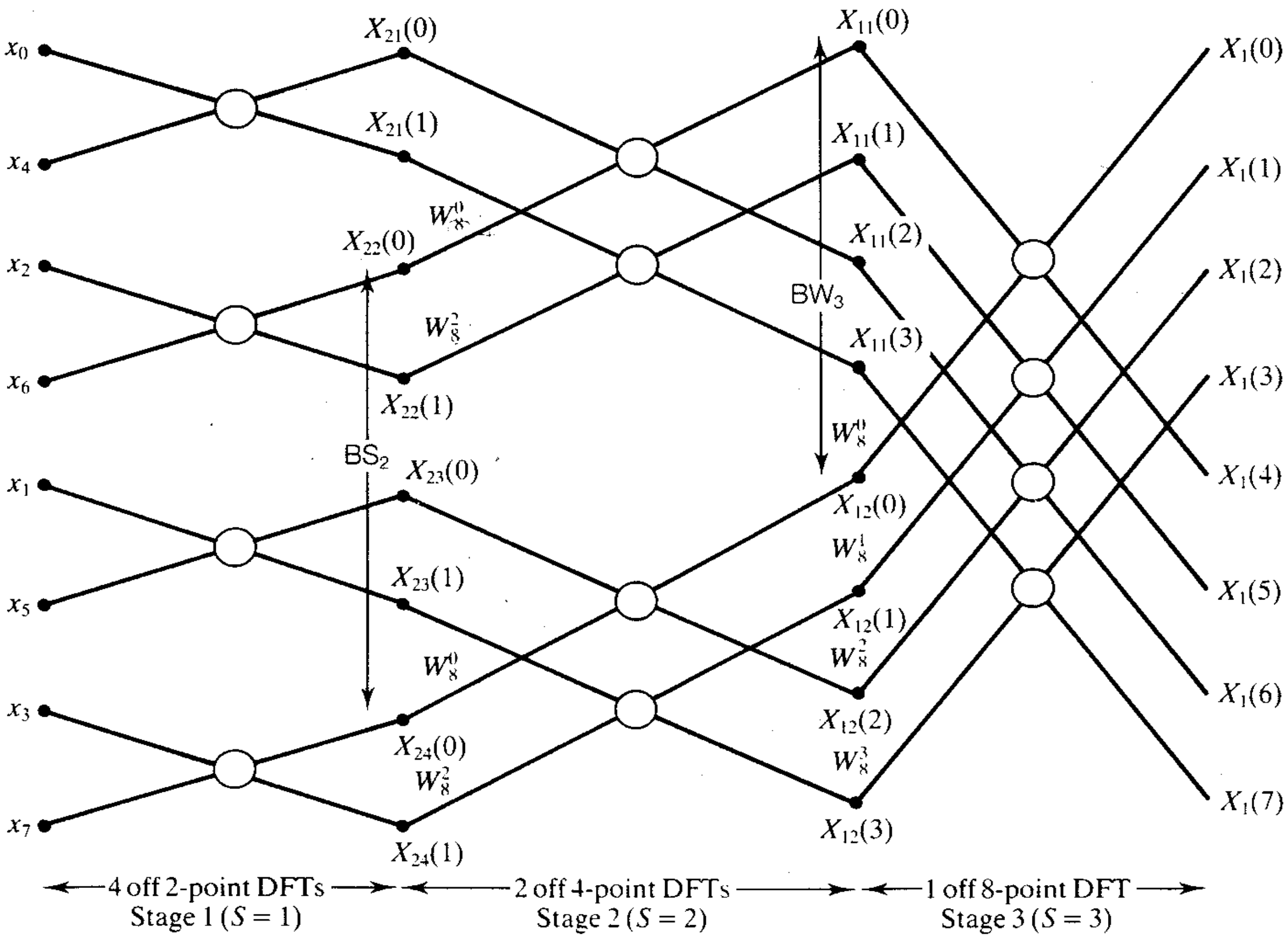
$$X_{22}(k) = x_2 + W_{N/4}^k x_6 \ ,$$

$$X_{23}(k) = x_1 + W_{N/4}^k x_5 \ ,$$

$$X_{24}(k) = x_3 + W_{N/4}^k x_7 \ .$$

A.5 Fast Fourier Transform

☞ **The illustration of FFT butterflies for an 8-point DFT**



A.5 Fast Fourier Transform

☞ **The comparisons of complexity between DFT and FFT**

N	DFT Number of complex multiplications	DFT Number of complex additions	FFT Number of complex multiplications	FFT Number of complex additions	Ratio of DFT multiplications to FFT multiplications	Ratio of DFT additions to FFT additions
2	4	2	1	2	4	1
4	16	12	4	8	4	1.5
8	64	56	12	24	5.3	2.3
16	256	240	32	64	8.0	3.75
32	1 024	992	80	160	12.8	6.2
64	4 096	4 032	192	384	21.3	10.5
128	16 384	16 256	448	896	36.6	18.1
256	65 536	65 280	1 024	2 048	64.0	31.9
512	262 144	261 632	2 304	4 608	113.8	56.8
1024	1 048 576	1 047 552	5 120	10 240	204.8	102.3
2048	4 194 304	4 192 256	11 264	22 528	372.4	186.1
4096	16 777 216	16 773 120	24 576	49 152	682.7	341.3
8192	67 108 864	67 100 672	53 248	106 496	1 260.3	630.0

射 频 和 天 线 设 计 培 训 课 程 推 荐

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课程网址：http://www.edatop.com/peixun/antenna/134.html

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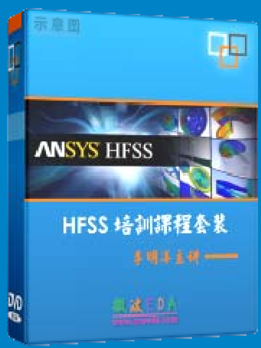
课程网址：http://www.edatop.com/peixun/cst/24.html



HFSS 学习培训课程套装

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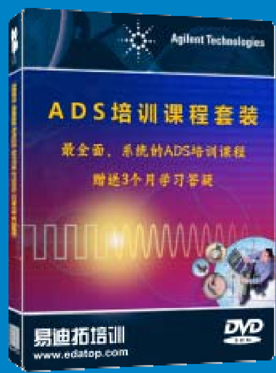

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※ 经验丰富的一线资深工程师讲授，结合实际工程案例，直观、实用、易学

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